

On the Brauer group of diagonal cubic surfaces

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Abstract

We are concerned with finding explicit generators of the Brauer group of diagonal cubic surfaces in terms of norm residue symbols, which was originally studied by Manin.

We introduce the notion of uniform generators and find that the Brauer group of some classes of diagonal cubic surfaces have uniform generators. However, we also prove that the Brauer group of general diagonal cubic surfaces do not have such ones. This reveals that a result of Manin for certain diagonal cubic surfaces cannot be generalized in some sense.

1 Introduction

Let k be a field of characteristic zero and containing a fixed primitive cubic root ζ of unity. In this paper, we study the cohomological Brauer group of diagonal cubic surfaces V over k , that is, smooth projective surfaces defined by a homogeneous equation of the form

$$x^3 + by^3 + cz^3 + dt^3 = 0,$$

where $b, c, d \in k^*$. In particular, we are concerned with the following two natural problems:

- (1) Determine the structure of $\text{Br}(V)$ as an abelian group.
- (2) Find generators of $\text{Br}(V)$ in terms of norm residue symbols.

In general, the Brauer group of a variety plays an important role in studying its arithmetic and its geometry. For applications to the Hasse principle, see for example, [11] and [17]. It is also used for studying zero-cycles ([10] and [2]). For applications to the rationality problem, see [1]. For such studies, we want to know in advance the structure and generators of its Brauer group.

For diagonal cubic surfaces, an original work in this direction was due to Manin [12]. He gave a complete answer to the above two problems for diagonal cubic surfaces of the form $x^3 + y^3 + z^3 + dt^3 = 0$ for $d \in k^* \setminus (k^*)^3$. Let $\pi: V \rightarrow \text{Spec } k$ be the structure morphism and put $\text{Br}(V)/\text{Br}(k) := \text{Br}(V)/\pi^* \text{Br}(k)$. In this case, $\text{Br}(V)/\text{Br}(k) \cong (\mathbb{Z}/3\mathbb{Z})^2$ and

$$\left\{ d, \frac{x + \zeta y}{x + y} \right\}_3, \quad \left\{ d, \frac{x + z}{x + y} \right\}_3$$

are its symbolic generators, where

$$\{\cdot, \cdot\}_3: K_2^M(k(V)) \rightarrow H^2(k(V), \mu_3^{\otimes 2}) \cong H^2(k(V), \mu_3) \hookrightarrow \text{Br}(k(V)),$$

is a norm residue symbol map. As an application of these symbolic generators, Saito and Sato [15] recently computed the degree-zero part of the Chow group of zero-cycles on such cubic surfaces over p -adic fields explicitly, even in the case $p = 3$.

In this paper, we study these problems in a more general setting where the equation of V is of the forms $x^3 + y^3 + cz^3 + dt^3 = 0$ and $x^3 + by^3 + cz^3 + dt^3 = 0$.

First, we prove the following theorem, which gives an answer to the problems (1) and (2) for the case $x^3 + y^3 + cz^3 + dt^3 = 0$.

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Theorem 1.1. *Let k be as above and V be the cubic surface over k defined by an equation $x^3 + y^3 + cz^3 + dt^3 = 0$, where c and $d \in k^*$. Assume that c , d , cd and d/c are not contained in $(k^*)^3$. Then we have the following:*

- (1) *The group $\text{Br}(V)/\text{Br}(k)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.*
- (2) *The element*

$$e_1 = \left\{ \frac{d}{c}, \frac{x + \zeta y}{x + y} \right\}_3 \in \text{Br}(k(V))$$

is contained in $\text{Br}(V)$.

- (3) *The image of e_1 in $\text{Br}(V)/\text{Br}(k)$ is a generator of this group.*

The claim (1) is essentially due to [3]. Recently Colliot-Thélène and Wittenberg found a symbolic generator of $\text{Br}(V)/\text{Br}(k)$ for $V : x^3 + y^3 + 2z^3 = at^3$ when k does not contain a primitive cubic root of unity ([4], Proposition 2.1). In this case, our symbolic generator was also appeared in the proof of this proposition.

We note that in the result of Manin and Theorem 1.1, we can take a generator *uniformly*. More precisely, let c and d be indeterminates, $F = k(c, d)$, V be the cubic surface $x^3 + y^3 + cz^3 + dt^3 = 0$ over F , and

$$e(c, d) = \left\{ \frac{d}{c}, \frac{x + \zeta y}{x + y} \right\}_3$$

be an element in $\text{Br}(V)$. Let $P = (c_0, d_0)$ be a point in $k^* \times k^*$ with c_0 , d_0 , $c_0 d_0$ and d_0/c_0 not contained in $(k^*)^3$, and V_P the surface defined by $x^3 + y^3 + c_0 z^3 + d_0 t^3 = 0$. If we want a symbolic generator of $\text{Br}(V_P)/\text{Br}(k)$, we can get it by specializing $e(c, d)$ at P . We denote this element by $\text{sp}(e(c, d); P)$. A precise definition of the specialization will be given in §2. In general, it is not necessary that the Brauer group of a given variety has such uniform generators.

Concerning the problem whether symbolic generators can be chosen uniformly or not, we prove a non-existence result as stated below. Let $F = k(b, c, d)$, where b, c, d are indeterminates over k , and let V be the projective cubic surface over F defined by the equation $x^3 + by^3 + cz^3 + dt^3 = 0$. For $P = (b_0, c_0, d_0) \in k^* \times k^* \times k^*$, let V_P be the projective cubic surface over k defined by the equation $x^3 + b_0 y^3 + c_0 z^3 + d_0 t^3 = 0$. For $e \in \text{Br}(V)$, we will define its specialization at P , $\text{sp}(e; P) \in \text{Br}(V_P)$. Put

$$\mathcal{P}_k = \{P \in (\mathbb{G}_{m,k})^3 \mid \text{Br}(V_P)/\text{Br}(k) \cong \mathbb{Z}/3\mathbb{Z}\}.$$

Note ([3]) that $\text{Br}(V_P)/\text{Br}(k)$ isomorphic to either of 0, $\mathbb{Z}/3\mathbb{Z}$ and $(\mathbb{Z}/3\mathbb{Z})^2$ and that Manin dealt with the last case, as stated before.

We first prove the following:

Theorem 1.2. *Let k, F and V be as above. Then*

$$\text{Br}(V)/\text{Br}(F) = 0.$$

Note that this vanishingness does not follow directly from a seven-term exact sequence induced by the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(F, H^q(\overline{V}, \mathbb{G}_m)) \Rightarrow H^{p+q}(V, \mathbb{G}_m)$$

since $E_2^{1,1} \neq 0$, $V(F) = \emptyset$ and $\text{cd}(F) \geq 3$. As far as we know, this would be the first example of computation of Brauer groups for such varieties.

As a corollary of Theorem 1.2, we can obtain the following non-existence result:

Corollary 1.3. *Let k, F and V as in Theorem 1.2. Assume moreover $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$. Then there is no element $e \in \text{Br}(V)$ satisfying the following condition:*

there exists a dense open subset $W \subset (\mathbb{G}_{m,k})^3$ such that $\text{sp}(e; \cdot)$ is defined on $W(k) \cap \mathcal{P}_k$ and for all $P \in W(k) \cap \mathcal{P}_k$, $\text{sp}(e; P)$ is a generator of $\text{Br}(V_P)/\text{Br}(k)$.

We note that the assumption $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$ is equivalent to the Zariski density of $\mathcal{P}_k \subset (\mathbb{G}_{m,k})^3$, which is essentially necessary to prove Theorem 1.2. We easily see that this assumption holds for various fields, for example, all finitely generated fields over $\mathbb{Q}(\zeta)$ and $\mathbb{Q}_p(\zeta)$ for any prime number p , and hence this is a mild assumption.

This paper is written in the following fashion. In §2, we describe the Brauer group of varieties in terms of Galois cohomology. We also define specialization of Brauer groups. In §3 we focus on diagonal cubic surfaces, especially their Picard groups and Galois action on them. In §4, we give a proof of Theorem 1.1. Finally, in §5, we prove Theorem 1.2 by computing the image under a differential $d^{1,1}$ appearing in the above spectral sequence explicitly. We also discuss the condition $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$ appearing in Corollary 1.3. Finally, we prove Corollary 1.3.

Notation. For a group A and $f \in \text{End}(A)$, we denote by ${}_f A$ the kernel of f .

Throughout this paper, all fields are of characteristic zero. For a field k , we denote a separable closure of k by \bar{k} . We fix such a field \bar{k} and each algebraic separable extension of k is always considered as a subfield of this \bar{k} . If k is a discrete valuation field, the field k^{ur} denotes the maximal unramified extension of k .

Fix a positive integer n and assume that k contains a primitive n -th root ζ_n of unity. For f and $g \in k^*$, we denote by $\{f, g\}_n$ the image of $f \otimes g$ under a usual norm residue symbol map

$$k^* \otimes k^* \rightarrow H^1(k, \mu_n) \otimes H^1(k, \mu_n) \xrightarrow{\cup} H^2(k, \mu_n^{\otimes 2}) \cong H^2(k, \mu_n) \cong {}_n \text{Br}(k),$$

where the third map is induced by $\zeta_n^i \otimes \zeta_n^j \mapsto \zeta_n^{ij}$.

For a scheme V , all cohomology groups of V mean étale cohomology.

2 Preliminaries of Brauer groups

Let k be a field and $\pi: V \rightarrow \text{Spec } k$ a variety over k . In this section, we see two descriptions of $\text{Br}(V)$ in terms of Galois cohomology of k . We also introduce the notion of specialization of Brauer groups.

First, we recall a fundamental exact sequence. By the Hochschild-Serre spectral sequence

$$H^p(k, H^q(\bar{V}, \mathbb{G}_m)) \Rightarrow H^{p+q}(V, \mathbb{G}_m),$$

we have the following exact sequence

$$0 \rightarrow \text{Br}_1(V)/\text{Br}(k) \rightarrow H^1(k, \text{Pic}(\bar{V})) \xrightarrow{d^{1,1}} H^3(k, \bar{k}^*), \quad (2.1)$$

where

$$\text{Br}_1(V) := \text{Ker}(\text{Br}(V) \rightarrow \text{Br}(\bar{V})), \quad \text{Br}_1(V)/\text{Br}(k) := \text{Br}_1(V)/\pi^* \text{Br}(k).$$

By this sequence, we know that $\text{Br}_1(V)/\text{Br}(k)$ has an inclusion into $H^1(k, \text{Pic}(\bar{V}))$. It is not clear whether this inclusion is an isomorphism or not. However, here are the following sufficient conditions:

Lemma 2.1. *Let V be a variety over a field k . If $\text{cd}(k) \leq 2$ or $V(k) \neq \emptyset$, then*

$$\text{Br}_1(V)/\text{Br}(k) \cong H^1(k, \text{Pic}(\bar{V})).$$

Proof. The first (resp. the second) assumption implies $H^3(k, \bar{k}^*) = 0$ (resp. $H^3(k, \bar{k}^*) \rightarrow H^3(V, \mathbb{G}_m)$ is injective), which implies the surjectivity of $\text{Br}_1(V)/\text{Br}(k) \rightarrow H^1(k, \text{Pic}(\bar{V}))$. \square

Secondly, we give another description of Brauer group of varieties. We use the following result in §4. We describe the following Brauer group

$$\text{Br}(V_L/V) := \text{Ker}(\text{Br}(V) \rightarrow \text{Br}(V_L)),$$

where L/k is a Galois extension. The claim is:

Proposition 2.2. *Let V be a smooth, geometrically integral variety over a field k and let L be a Galois extension of k . Put $G := \text{Gal}(L/k)$. Then we have an exact sequence*

$$0 \rightarrow \text{Br}(V_L/V) \rightarrow H^2(G, L(V)^*) \xrightarrow{\text{div}} H^2(G, \text{Div}(V_L)),$$

where div is naturally induced by

$$\text{div}: L(V)^* \rightarrow \text{Div}(V_L); \quad f \mapsto \text{div}(f).$$

Proof. Let $j: \eta = \text{Spec } k(V) \rightarrow V$ be the generic point of V . We have the following exact sequence of étale sheaves on V :

$$0 \rightarrow \mathbb{G}_m \rightarrow j_* \mathbb{G}_{m,\eta} \rightarrow \text{Div}_V \rightarrow 0, \quad (2.2)$$

where Div_V is the sheaf of Cartier divisors on V . By regularity of V , we have $H^1(V, \text{Div}_V) = 0$. Moreover, we have $H^i(V, j_* \mathbb{G}_{m,\eta}) \cong H^i(k(V), \mathbb{G}_m)$ for all $i \geq 0$. These yield the commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Br}(V) & \longrightarrow & \text{Br}(k(V)) & \longrightarrow & H^2(V, \text{Div}_V) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Br}(V_L) & \longrightarrow & \text{Br}(L(V)) & \longrightarrow & H^2(V_L, \text{Div}_V). \end{array}$$

Taking the kernel of each column, we obtain the exact sequence

$$0 \rightarrow \text{Br}(V_L/V) \rightarrow \text{Br}(L(V)/k(V)) \rightarrow \text{Ker}(H^2(V, \text{Div}_V) \rightarrow H^2(V_L, \text{Div}_V)).$$

Applying the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G, H^q(V_L, \cdot)) \Rightarrow E^{p+q} = H^{p+q}(V, \cdot)$$

to sheaves $j_* \mathbb{G}_{m,\eta} \rightarrow \text{Div}_V$ on V , we have the following commutative diagram

$$\begin{array}{ccc} H^2(G, L(V)^*) & \xrightarrow{\cong} & \text{Ker}(\text{Br}(k(V)) \rightarrow \text{Br}(L(V))) \\ \downarrow & & \downarrow \\ H^2(G, \text{Div}(V_L)) & \xrightarrow{\cong} & \text{Ker}(H^2(V, \text{Div}_V) \rightarrow H^2(V_L, \text{Div}_V)), \end{array}$$

which completes the proof of Proposition 2.2. \square

Finally, in the last of this section, we introduce the notion of specialization of Brauer groups. In §4, we will see that the Brauer group of surfaces of the form $x^3 + y^3 + cz^3 + dt^3 = 0$ has a *uniform* symbolic generator, that is, if we put

$$e(c, d) = \left\{ \frac{d}{c}, \frac{x + \zeta y}{x + y} \right\}_3$$

where c and d are considered as *indeterminates* and if we want a symbolic generator of $\text{Br}(V_P)/\text{Br}(k)$, where V_P is the surface of the form $x^3 + y^3 + c_0 z^3 + d_0 t^3 = 0$ with c_0 and $d_0 \in k^*$, we can get it by specializing $e(c, d)$ at $(c, d) = (c_0, d_0)$.

To make this notion of uniformity precise, we define specialization as follows. Let k be a field, \mathcal{O}_F a polynomial ring over k with r variables, F its fractional field and f_1, \dots, f_m homogeneous polynomials in $\mathcal{O}_F[x_0, \dots, x_n]$. Let \mathcal{V} be the projective scheme over \mathcal{O}_F defined as:

$$\mathcal{V} = \text{Proj}(\mathcal{O}_F[x_0, \dots, x_n]/(f_1, \dots, f_m)) \xrightarrow{\pi} \text{Spec } \mathcal{O}_F.$$

Let $\pi_F: V := \mathcal{V}_F \rightarrow \text{Spec } F$ be the base change of π to $\text{Spec } F$. Assume that V is smooth over F . Let $e \in \text{Br}(V)$ be an arbitrary element. If $(S_i)_{i \in I}$ is the projective system of the non-empty affine open subschemes in $\mathbb{A}_k^r = \text{Spec } \mathcal{O}_F$, we have

$$\text{proj lim}_i (\mathcal{V} \times_{\mathbb{A}_k^r} S_i) \cong V,$$

and there exists a non-empty affine open subscheme S and $\tilde{e} \in \text{Br}(\mathcal{V} \times_{\mathbb{A}_k^r} S)$ satisfying that $\mathcal{V} \times_{\mathbb{A}_k^r} S$ is smooth over S and that

$$\text{res}_{\text{Spec } F}^S(\tilde{e}) = e,$$

where $\text{res}_{\text{Spec } F}^S: \text{Br}(\mathcal{V} \times_{\mathbb{A}_k^r} S) \rightarrow \text{Br}(V)$. This follows from [7](Proposition 17.7.8) and [14](III, Lemma 1.16). For a given $P \in S(k)$, we have the following diagram:

$$\begin{array}{ccccc} V_P & \xrightarrow{P} & \mathcal{V} \times_{\mathbb{A}_k^r} S & \xleftarrow{\quad} & V \\ \downarrow \pi_0 & & \square & & \downarrow \pi_S \\ \text{Spec } k & \xrightarrow{P} & S & \xleftarrow{\quad} & \text{Spec } F \end{array}$$

where $V_P := \mathcal{V} \times_{\mathbb{A}_k^r} \text{Spec } k$. We define the specialization of e at P as

$$\text{sp}(e; P) := P^* \tilde{e} \in \text{Br}(V_P).$$

By the regularity of $\mathcal{V} \times_{\mathbb{A}_k^r} S$, the map $\text{res}_{\text{Spec } F}^S$ is injective, which implies that this definition is independent of the choice of S .

3 Preliminaries of diagonal cubic surfaces

In this section, we are concerned with diagonal cubic surfaces, in particular, their Picard groups and their Galois structures. We mainly use the same notation as in [3]. Let k be a field containing a primitive cubic root ζ of unity. Let V be the projective surface over k defined by a homogeneous equation

$$ax^3 + by^3 + cz^3 + dt^3 = 0,$$

where a, b, c and d are in k^* . Let $\pi: V \rightarrow \text{Spec } k$ denote the structure morphism. Now we put

$$\lambda = \frac{b}{a}, \quad \mu = \frac{c}{a} \quad \text{and} \quad \nu = \frac{ad}{bc},$$

and then we can write as the equation of V

$$x^3 + \lambda y^3 + \mu z^3 + \lambda \mu \nu t^3 = 0.$$

We define some extensions of k which are frequently used in this paper. Let α, α' and γ be solutions in \bar{k} of equations $X^3 - \lambda = 0$, $X^3 - \mu = 0$ and $X^3 - \nu = 0$ respectively. Put $\beta = \alpha\gamma$ and $\beta' = \alpha'\gamma$. We define a field k' and k'' as $k(\alpha, \gamma)$ and $k'(\alpha')$.

\bar{V} is a del Pezzo surface, obtained from \mathbb{P}^2 by blowing-up general 6 points. Thus $\text{Pic}(\bar{V})$ is free of rank 7. We can write 27 lines on \bar{V} explicitly:

$$\begin{aligned} L(i): x + \zeta^i \alpha y &= z + \zeta^i \beta t = 0, \\ L'(i): x + \zeta^i \alpha y &= z + \zeta^{i+1} \beta t = 0, \\ L''(i): x + \zeta^i \alpha y &= z + \zeta^{i+2} \beta t = 0, \\ M(i): x + \zeta^i \alpha' z &= y + \zeta^{i+1} \beta' t = 0, \\ M'(i): x + \zeta^i \alpha' z &= y + \zeta^{i+2} \beta' t = 0, \\ M''(i): x + \zeta^i \alpha' z &= y + \zeta^i \beta' t = 0, \\ N(i): x + \zeta^i \alpha \beta' t &= y + \zeta^{i+2} \alpha^{-1} \alpha' z = 0, \\ N'(i): x + \zeta^i \alpha \beta' t &= y + \zeta^i \alpha^{-1} \alpha' z = 0, \\ N''(i): x + \zeta^i \alpha \beta' t &= y + \zeta^{i+1} \alpha^{-1} \alpha' z = 0, \end{aligned} \tag{3.1}$$

where i is either 0, 1 or 2.

Since six lines $L(0)$, $L(1)$, $L(2)$, $M(0)$, $M(1)$ and $M(2)$ are mutually skew, we can get a \bar{k} -morphism $\pi: \bar{V} \rightarrow \mathbb{P}^2$ by blowing down these six lines. We define $l \in \text{Pic}(\bar{V})$ as the inverse image of a line on \mathbb{P}^2 . Then we can obtain generators of $\text{Pic}(\bar{V}) \cong \mathbb{Z}^7$:

$$[L(0)], [L(1)], [L(2)], [M(0)], [M(1)], [M(2)], \text{ and } l, \quad (3.2)$$

where $[D]$ denotes the class of $D \in \text{Div}(\bar{V})$ in $\text{Pic}(\bar{V})$. Let H be the hyperplane section of \bar{V} defined by the equation $x = 0$,

$$[L] = [L(0)] + [L(1)] + [L(2)], \text{ and } [M] = [M(0)] + [M(1)] + [M(2)],$$

we have the following relation:

$$[H] = 3l - [L] - [M]. \quad (3.3)$$

We have $\text{Pic}(V_{k''}) \cong \text{Pic}(\bar{V})^{G_{k''}} \cong \mathbb{Z}^7$. As its generators, we take the classes corresponding to $[L(i)]$, $[M(i)]$ and $l \in \text{Pic}(\bar{V})$. By abuse of notation, we use the same symbols as in the case of $\text{Pic}(\bar{V})$.

By the geometrical rationality of V and the birational invariance of Brauer groups ([8], III, Théorème 7.1), we have $\text{Br}(\bar{V}) \cong \text{Br}(\mathbb{P}^2) = 0$. Hence we rewrite the sequence (2.1) as follows:

$$0 \rightarrow \text{Br}(V)/\text{Br}(k) \rightarrow H^1(k, \text{Pic}(\bar{V})) \xrightarrow{d^{1,1}} H^3(k, \bar{k}^*) \quad (\text{exact}). \quad (3.4)$$

This sequence plays a fundamental role in this paper.

The structure of the group $H^1(k, \text{Pic}(\bar{V}))$ is well-known:

Proposition 3.1 ([3], Proposition 1.).

$$H^1(k, \text{Pic}(\bar{V})) \cong \begin{cases} 0 & \text{if one of } \nu, \nu/\lambda, \nu/\mu \text{ is a cube in } k^*, \\ (\mathbb{Z}/3\mathbb{Z})^2 & \text{if exactly three of } \lambda, \mu, \lambda/\mu, \lambda\mu\nu, \lambda\nu, \mu\nu \\ & \text{are cubes in } k^*, \\ \mathbb{Z}/3\mathbb{Z} & \text{otherwise.} \end{cases}$$

In [3], this proposition is proved by showing the following isomorphism

$$H^1(k'/k, \text{Pic}(V_{k'})) \cong H^1(k, \text{Pic}(\bar{V}))$$

and reducing to explicit calculation of cohomology of the finite extension k'/k with coefficients $\text{Pic}(V_{k'})$. Since $V(k') \neq \emptyset$, we have $\text{Pic}(V_{k'}) \cong \text{Pic}(V_{k''})^{\text{Gal}(k''/k')}$; moreover using the explicit defining equations of divisors (3.1) and the equation (3.3), we see that

$$\text{Pic}(V_{k'}) = \mathbb{Z}l \oplus \mathbb{Z}[L(0)] \oplus \mathbb{Z}[L(1)] \oplus \mathbb{Z}[L(2)] \oplus \mathbb{Z}[M]. \quad (3.5)$$

In the following, we give an explicit generating cocycle of $H^1(k'/k, \text{Pic}(V_{k'}))$ under some conditions stated below.

First, in §4, we consider cubic surfaces over k defined by $x^3 + y^3 + cz^3 + dt^3 = 0$. If one of cd and d/c is in $(k^*)^3$, we have $H^1(k, \text{Pic}(\bar{V})) = 0$ by Proposition 3.1 and hence $\text{Br}(V)/\text{Br}(k) = 0$. Therefore we need not consider this case. If neither cd nor d/c is in $(k^*)^3$ and c is in $(k^*)^3$, such surfaces are isomorphic to surfaces defined by $x^3 + y^3 + z^3 + dt^3 = 0$ and Manin has already found the structure and the generators of their Brauer groups. Hence we may assume c, d, cd and $d/c \notin (k^*)^3$.

Secondly, in §5, we consider surfaces $x^3 + by^3 + cz^3 + dt^3 = 0$ over $k(b, c, d)$, where b, c and d are indeterminates. Hence $k(\alpha, \gamma, \alpha')/k$ is an extension of degree 27.

Therefore in the sequel of this section, we always assume one of the following conditions:

- (i) $\lambda = 1$, and neither $\mu, \nu, \mu\nu$ nor ν/μ is cubic in k^* .
- (ii) $k(\alpha, \gamma, \alpha')$ is a field extension of k with degree 27.

By Proposition 3.1, $H^1(k'/k, \text{Pic}(V_{k'})) \cong \mathbb{Z}/3\mathbb{Z}$ in both cases. Note that under the condition (i), $k(\alpha) = k$ and hence k'/k is of degree 3.

Let s be the generator of $G = \text{Gal}(k'/k(\alpha))$ such that $s\gamma = \zeta\gamma$ and w the generator of $\text{Gal}(k''/k')$ such that $w\alpha' = \zeta\alpha'$. Note that G and $\text{Gal}(k''/k')$ are isomorphic to $\mathbb{Z}/3\mathbb{Z}$ under one of the above assumptions and such elements s and w do exist. Moreover, under the condition (ii), let t be the generator of $\text{Gal}(k'/k(\gamma))$ such that $t\alpha = \zeta\alpha$.

The claim is the following:

Proposition 3.2. (1) As a generator of $H^1(G, \text{Pic}(V_{k'}))$, we can take a class of the following cocycle ϕ' :

$$\phi'(1) = 0, \quad \phi'(s) = [L(0)] - [L(2)], \quad \phi'(s^2) = [L(0)] - [L(1)].$$

(2) Under the condition (ii), as a generator of $H^1(k'/k, \text{Pic}(V_{k'}))$, we can take a class of the following cocycle ϕ :

$$\phi((st)^i) = 0, \quad \phi(s(st)^i) = [L(0)] - [L(2)], \quad \phi(s^2(st)^i) = [L(0)] - [L(1)],$$

where i takes on any values in $\{0, 1, 2\}$.

Proof. (1) Since G is a finite cyclic group, we have the following isomorphism:

$$H^1(G, \text{Pic}(V_{k'})) \cong \hat{H}^{-1}(G, \text{Pic}(V_{k'})) \cong \frac{N_G \text{Pic}(V_{k'})}{I_G(\text{Pic}(V_{k'}))},$$

where the norm N_G maps x to $\sum_{s \in G} sx$ and the map I_G maps x to $sx - x$ for all $x \in \text{Pic}(V_{k'})$. The action of s on $\text{Pic}(V_{k'})$ is as follows:

$$s \begin{pmatrix} l \\ [L(0)] \\ [L(1)] \\ [L(2)] \\ M \end{pmatrix} = \begin{pmatrix} 4 & -1 & -1 & -1 & 2 \\ 2 & -1 & -1 & 0 & -1 \\ 2 & 0 & -1 & -1 & -1 \\ 2 & -1 & 0 & -1 & -1 \\ 3 & 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} l \\ [L(0)] \\ [L(1)] \\ [L(2)] \\ M \end{pmatrix}.$$

By a straightforward calculation, we know that $N_G \text{Pic}(V_{k'})/I_G(\text{Pic}(V_{k'})) \cong \mathbb{Z}/3\mathbb{Z}$ and $[L(1)] - [L(0)]$ is its generator. Computing the above isomorphisms explicitly, we easily find that ϕ' is a cocycle whose class in $H^1(G, \text{Pic}(V_{k'}))$ is a generator.

Next we consider the claim (2). First, noting that $[L(1)] - [L(0)]$ is st -invariant, we can easily check ϕ is a cocycle. Moreover, the image of $[\phi]$ under the restriction

$$H^1(k'/k, \text{Pic}(V_{k'})) \rightarrow H^1(G, \text{Pic}(V_{k'})) \cong \mathbb{Z}/3\mathbb{Z}$$

is the class $[\phi']$ appearing in (1). Hence ϕ is also a non-zero element, in fact, a generator of $H^1(k'/k, \text{Pic}(V_{k'}))$. \square

In the last of this section, we introduce the following description of $\text{Pic}(V_{k'})$, which plays an important role in §4 and §5. Let \mathcal{D} be the following free abelian group of rank 10:

$$\mathcal{D} = \mathbb{Z}H \oplus \bigoplus_{i=0}^2 \mathbb{Z}L(i) \oplus \bigoplus_{i=0}^2 \mathbb{Z}L'(i) \oplus \bigoplus_{i=0}^2 \mathbb{Z}L''(i).$$

We see that \mathcal{D} is a G -submodule of $\text{Div}(V_{k'})$ by using (3.1). Let \mathcal{D}_0 be the G -submodule generated by the following five divisors:

$$\begin{aligned} D_1 &= \text{div}(f_1) = (L(0) + L'(0) + L''(0)) - H, \\ D_2 &= \text{div}(f_2) = (L(1) + L'(1) + L''(1)) - H, \\ D_3 &= \text{div}(f_3) = (L(0) + L'(2) + L''(1)) - H, \\ D_4 &= \text{div}(f_4) = (L(1) + L'(0) + L''(2)) - H, \\ D_5 &= \text{div}(f_5) = (L(2) + L'(1) + L''(0)) - H, \end{aligned}$$

where

$$f_1 = \frac{x + \alpha y}{x}, \quad f_2 = \frac{x + \zeta \alpha y}{x}, \quad f_3 = \frac{z + \beta t}{x}, \quad f_4 = \frac{z + \zeta \beta t}{x}, \quad f_5 = \frac{z + \zeta^2 \beta t}{x}.$$

Lemma 3.3. *Let \mathcal{D} and \mathcal{D}_0 be as above. Then we have the following exact sequence of G -modules:*

$$0 \rightarrow \mathcal{D}_0 \rightarrow \mathcal{D} \rightarrow \text{Pic}(V_{k'}) \rightarrow 0.$$

Proof. For the exactness at $\text{Pic}(V_{k'})$, it suffices to show that we can write the classes l and $[M]$ as linear combinations of $[H]$, $[L(i)]$, $[L'(i)]$ and $[L''(i)]$. The intersection matrix with respect to the basis in (3.2) is the diagonal matrix with entries $-1, -1, -1, -1, -1, -1$ and 1 . By using this matrix, (3.1) and (3.3), we can write $[L'(0)]$ and $[L''(0)]$ as follows:

$$[L'(0)] = 2l - [L(0)] - [L(1)] - [M], \quad [L''(0)] = l - [L(0)] - [L(2)],$$

which implies the surjectivity of $\mathcal{D} \rightarrow \text{Pic}(V_{k'})$.

The exactness at \mathcal{D}_0 is trivial by definition and we also prove the exactness at \mathcal{D} by comparing the ranks of $\mathcal{D}_0, \mathcal{D}$ and $\text{Pic}(V_{k'})$. This completes the proof of this lemma. \square

4 The case $x^3 + y^3 + cz^3 + dt^3 = 0$

In this section, let k be a field containing a primitive cubic root ζ of unity. The result in this section is:

Theorem 4.1. *Let V be the cubic surface over k defined by a homogeneous equation $x^3 + y^3 + cz^3 + dt^3 = 0$, where c and $d \in k^*$. Moreover, we assume the condition (i) in §3, that is, c, d, cd and d/c are not in $(k^*)^3$. Then we have the following:*

- (1) *The group $\text{Br}(V)/\text{Br}(k)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.*
- (2) *The symbol*

$$e_1 = \left\{ \frac{d}{c}, \frac{x + \zeta y}{x + y} \right\}_3 \in \text{Br}(k(V))$$

is contained in $\text{Br}(V)$.

- (3) *The image of e_1 in $\text{Br}(V)/\text{Br}(k)$ is a generator of this group.*

Proof. Let $G = \text{Gal}(k'/k)$ and $s \in G$ a generator such that $s\gamma = \zeta\gamma$.

First we consider (1). This surface has a k -rational point $P = (1 : -1 : 0 : 0)$. Therefore the claim of (1) follows from Lemma 2.1 and Proposition 3.1.

Next we consider (2). Let ϕ' be as in Proposition 3.2 (1). Computing the cocycle $\partial'\phi'$, where

$$\partial': H^1(G, \text{Pic}(V_{k'})) \rightarrow H^2(G, k'(V)^*/k'^*),$$

is the connecting homomorphism induced by the exact sequence of G -modules:

$$0 \rightarrow k'(V)^*/k'^* \rightarrow \text{Div}(V_{k'}) \rightarrow \text{Pic}(V_{k'}) \rightarrow 0, \quad (4.1)$$

we can show

$$\partial'\phi'(s^i, s^j) = \left(\frac{f_2}{f_1} \right)^{a(i,j)} \in k'(V)^*/k'^*, \quad a(i,j) := \left\lfloor \frac{i+j}{3} \right\rfloor - \left\lfloor \frac{i}{3} \right\rfloor - \left\lfloor \frac{j}{3} \right\rfloor.$$

On the other hand, the symbol $\{\nu, f_2/f_1\}_3 \in \text{Br}(k(V))$ is equal to

$$(\chi_{3,\nu}, f_2/f_1) \in H^2(k(V), \overline{k(V)}^*),$$

where $\chi_{3,\nu} \in \text{Hom}(G_{k(V)}, \mathbb{Q}/\mathbb{Z}) \cong H^2(k(V), \mathbb{Z})$ is the cyclic character of order 3 associated to ν and (\cdot, \cdot) be the symbol defined by

$$(\cdot, \cdot): H^2(k(V), \mathbb{Z}) \otimes H^0(k(V), \overline{k(V)}^*) \xrightarrow{\cup} H^2(k(V), \overline{k(V)}^*); \quad (\chi, f) \mapsto f \cup \chi.$$

For a proof, see [16]. Moreover, by the following commutative diagram

$$\begin{array}{ccccc}
H^2(k(V), \mathbb{Z}) \otimes H^0(k(V), \overline{k(V)}^*) & \xrightarrow{\cup} & H^2(k(V), \overline{k(V)}^*) \\
\uparrow & & \uparrow \\
H^2(k'(V)/k(V), \mathbb{Z}) \otimes H^0(k'(V)/k(V), k'(V)^*) & \xrightarrow{\cup} & H^2(k'(V)/k(V), k'(V)^*) \\
\cong \uparrow & & \cong \uparrow \\
H^2(G, \mathbb{Z}) \otimes H^0(G, k'(V)^*) & \xrightarrow{\cup} & H^2(G, k'(V)^*),
\end{array}$$

$(\chi_\nu, f_2/f_1)$ can be considered as an element in $H^2(G, k'(V)^*)$, and we see that the corresponding cocycle is of the form

$$\left\{ \nu, \frac{f_2}{f_1} \right\}_3 (s^i, s^j) = \left(\frac{f_2}{f_1} \right)^{a(i,j)} \in k'(V)^*.$$

Finally, we have the following commutative diagram with all rows and columns exact:

$$\begin{array}{ccccccc}
& \text{Br}(k'/k) & \xlongequal{\quad} & \text{Br}(k'/k) & & & \\
& \downarrow & & \downarrow & & & \\
0 & \longrightarrow & \text{Br}(V_{k'}/V) & \longrightarrow & H^2(G, k'(V)^*) & \xrightarrow{\text{div}} & H^2(G, \text{Div}(V_{k'})) \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & H^1(G, \text{Pic}(V_{k'})) & \xrightarrow{\partial'} & H^2(G, k'(V)^*/k'^*) & \xrightarrow{\text{div}} & H^2(G, \text{Div}(V_{k'})),
\end{array}$$

where the middle column is induced by the exact sequence

$$0 \rightarrow k'^* \rightarrow k'(V)^* \rightarrow k'(V)^*/k'^* \rightarrow 0,$$

the middle row is the result of Proposition 2.2, the bottom row is induced by the exact sequence (4.1) and the triviality of its leftmost term follows from the fact that the action of G on $\text{Div}(V_{k'})$ maps one basis to another. Then the map

$$\text{Br}(V_{k'}/V) \rightarrow H^1(G, \text{Pic}(V_{k'}))$$

is naturally induced by

$$H^2(G, k'(V)^*) \rightarrow H^2(G, k'(V)^*/k'^*).$$

The fact that $\partial'\phi'$ and $\{\nu, f_2/f_1\}_3$ coincide in $H^2(G, k'(V)^*/k'^*)$ shows that

$$\{\nu, f_2/f_1\}_3 = \left\{ \frac{d}{c}, \frac{x + \zeta y}{x + y} \right\}_3 \in \text{Br}(V_{k'}/V) \subset \text{Br}(V),$$

which completes the proof of (2).

Finally we consider (3). By the above argument, $[\phi']$ and $\{\nu, f_2/f_1\}_3$ coincide in $H^1(G, \text{Pic}(V_{k'}))$. Hence we can take

$$\left\{ \frac{d}{c}, \frac{x + \zeta y}{x + y} \right\}_3$$

as a generator of the group $\text{Br}(V)/\text{Br}(k)$. This completes the proof of Theorem 4.1. \square

By using the specialization of Brauer groups, we can formulate Theorem 4.1 as follows:

Corollary 4.2. *Let k be as in Theorem 4.1, $\mathcal{O}_F = k[c, d]$, F its fractional field and*

$$V = \text{Proj}(F[x, y, z, t]/(x^3 + y^3 + cz^3 + dt^3)).$$

Then

$$e_1 = \left\{ \frac{d}{c}, \frac{x + \zeta y}{x + y} \right\}_3 \in \text{Br}(V)$$

is a uniform generator, that is, for all $P = (c_0, d_0) \in k^* \times k^*$ such that $c_0, d_0, c_0 d_0$ and $d_0/c_0 \notin (k^*)^3$, $\text{sp}(e_1; P)$ is a generator of $\text{Br}(V_P)/\text{Br}(k)$.

Proof. We confirm that $\text{sp}(e_1; P)$ is in fact the desired symbol, that is,

$$\left\{ \frac{d_0}{c_0}, \frac{x + \zeta y}{x + y} \right\}_3.$$

We define $S \subset \mathbb{A}_k^2$ and \mathcal{V} to be:

$$\begin{aligned} S &= \mathbb{G}_{m,k} \times \mathbb{G}_{m,k} = \text{Spec } k[c^\pm, d^\pm], \\ \mathcal{V} &= \text{Proj } \mathcal{O}_F[x, y, z, t]/(x^3 + y^3 + cz^3 + dt^3). \end{aligned}$$

where the symbol c^\pm is the abbreviation of c and c^{-1} . Put $\mathcal{V} \times S := \mathcal{V} \times_{\mathbb{A}_k^2} S$. we see that $\mathcal{V} \times S$ is smooth over S . Moreover, we note that e_1 can lift to $\tilde{e} \in \text{Br}(\mathcal{V} \times S)$. This follows from concrete calculations using the following exact sequence, which is a consequence of the absolute purity due to Gabber [5]:

$$0 \rightarrow \text{Br}(\mathcal{V} \times S) \rightarrow \text{Br}(F(V)) \xrightarrow{\oplus_x \text{res}_x} \bigoplus H^1(\kappa(x), \mathbb{Q}/\mathbb{Z}),$$

where the sum is taken over all points x of codimension one in $\mathcal{V} \times S$, and $\kappa(x)$ is the residue field of x . Hence we can use the above S and \tilde{e} to construct the specialization of e_1 .

Now we define the subscheme U of $\mathcal{V} \times S$ as follows:

$$U = \mathcal{V} \times S \setminus (D_+(x + y) \cup D_+(x + \zeta y)),$$

where $D_+(f)$ is the non-vanishing locus of a homogeneous polynomial f . Explicitly, if we put

$$R := \frac{k[c^\pm, d^\pm] \left[\frac{x}{x+y}, \frac{y}{x+y}, \frac{z}{x+y}, \frac{t}{x+y}, \frac{x+y}{x+\zeta y} \right]}{\left(\frac{x^3 + y^3 + cz^3 + dt^3}{(x+y)^3} \right)},$$

then $U = \text{Spec } R$. We have

$$\frac{d}{c}, \frac{x + \zeta y}{x + y} \in \Gamma(U, \mathcal{O}_U)^*$$

and hence

$$\left\{ \frac{d}{c}, \frac{x + \zeta y}{x + y} \right\}_3 \in H^2(U, \mu_3),$$

where

$$\{\cdot, \cdot\}_3: \Gamma(U, \mathcal{O}_U)^* \otimes \Gamma(U, \mathcal{O}_U)^* \rightarrow H^1(U, \mu_3) \otimes H^1(U, \mu_3) \xrightarrow{\cup} H^2(U, \mu_3^{\otimes 2}) \cong H^2(U, \mu_3)$$

is norm residue map defined similarly as in the field case. Take any $P = (c_0, d_0) \in k^* \times k^*$ such that $c_0, d_0, c_0 d_0$ and $c_0/d_0 \notin (k^*)^3$ and put $R_P := R/(c - c_0, d - d_0)$. We have the canonical morphism $P^*: U \rightarrow U_P := \text{Spec } R_P$ and the following commutative diagram:

$$\begin{array}{ccccc} \text{Br}(U_P) & \xleftarrow{P^*} & \text{Br}(U) & \longrightarrow & \text{Br}(F(V)) \\ \uparrow \text{res}_{U_P}^{V_P} & & \uparrow \text{res}_U^{\mathcal{V} \times S} & & \uparrow \text{res}_{F(V)}^V \\ \text{Br}(V_P) & \xleftarrow{P^*} & \text{Br}(\mathcal{V} \times S) & \longrightarrow & \text{Br}(V) \end{array}$$

Therefore we get

$$\begin{aligned}
\text{res}_{U_P}^{V_P}(\text{sp}(e_1; P)) &= \text{res}_{U_P}^{V_P}(P^*(\tilde{e})) \\
&= P^*(\text{res}_U^{\mathcal{V} \times S}(\tilde{e})) \\
&= P^*\left(\left\{\frac{d}{c}, \frac{x + \zeta y}{x + y}\right\}_3\right) \\
&= \left\{\frac{d_0}{c_0}, \frac{x + \zeta y}{x + y}\right\}_3
\end{aligned}$$

and complete the proof. \square

5 The case $x^3 + by^3 + cz^3 + dt^3 = 0$

In this section, let k be a field containing a primitive cubic root ζ of unity, λ, μ and ν indeterminates, $\mathcal{O}_F = k[\lambda, \mu, \nu]$, $F = k(\lambda, \mu, \nu)$ and

$$\mathcal{V} = \text{Proj}(\mathcal{O}_F[x, y, z, t]/(x^3 + \lambda y^3 + \mu z^3 + \lambda \mu \nu t^3)).$$

Put $V = \mathcal{V} \times_{\mathbb{A}_k^3} \text{Spec } F$. For all $P \in (\mathbb{G}_{m,k})^3(k)$, $V_P = \mathcal{V} \times_{\mathbb{A}_k^3} \text{Spec } k(P)$ is smooth over k . In this section, we are mainly devoted to proving the following:

Theorem 5.1.

$$\text{Br}(V)/\text{Br}(F) = 0.$$

As a corollary of this result, we obtain the non-existence of uniform generators. We define the set \mathcal{P}_k to be

$$\{P \in (\mathbb{G}_{m,k})^3(k) \mid \text{Br}(V_P)/\text{Br}(k) \cong \mathbb{Z}/3\mathbb{Z}\}.$$

Here we have the following

Proposition 5.2. *The following conditions are equivalent:*

- (1) \mathcal{P}_k is Zariski dense in $(\mathbb{G}_{m,k})^3$;
- (2) \mathcal{P}_k is non-empty;
- (3) $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$.

We define $C(k)$ to be the above equivalent conditions. We easily see that the condition (3) holds for various fields, for example,

- any finitely generated field over $\mathbb{Q}(\zeta)$ or $\mathbb{Q}_p(\zeta)$ for any prime number p ;
- a function field of any variety over \mathbb{C} or \mathbb{R} of dimension ≥ 1 .

Thus this condition $C(k)$ is mild and reasonable.

Now we can state the following non-existence result:

Corollary 5.3. *Let k and V be as above. Assume moreover $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$. Then there is no element $e \in \text{Br}(V)$ satisfying the following property:*

there exists a dense open subset $W \subset (\mathbb{G}_{m,k})^3$ such that $\text{sp}(e; \cdot)$ is defined on $W(k) \cap \mathcal{P}_k$ and for all $P \in W(k) \cap \mathcal{P}_k$, $\text{sp}(e; P)$ is a generator of $\text{Br}(V_P)/\text{Br}(k)$.

Remark 5.4. Let V be a cubic surface over k of the form $x^3 + by^3 + cz^3 + dt^3 = 0$. Assume $H^1(k, \text{Pic}(\overline{V})) = \mathbb{Z}/3\mathbb{Z}$ and $V(k) = \emptyset$. We note some known results of the structure and generators of $\text{Br}(V)/\text{Br}(k)$.

1. By a theorem of Merkurjev-Suslin [13], we always write a generator of $\text{Br}(V)/\text{Br}(k)$ as a sum of norm residue symbols.

2. If $\text{cd}(k) \leq 2$, $\text{Br}(V)/\text{Br}(k)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Its symbolic generator is not “uniform” by Corollary 5.3 and we do not know this can be written by *one* symbol $\{f, g\}_3$ for some $f, g \in k(V)^*$. However, here is a partial result. Let k be a field satisfying the following condition:

For any cubic extension L of k , the restriction $\text{Br}(k) \rightarrow \text{Br}(L)$ is surjective.

Some examples of k are

- a field k with $\text{cd}(k) \leq 1$.
- a local field k .

Then we find that a generator can be taken as $\{d/bc, f\}_3$ for some $f \in k(V)^*$.

3. If $\text{cd}(k) \geq 3$, it is difficult to determine whether $\text{Br}(V)/\text{Br}(k)$ is isomorphic to 0 or $\mathbb{Z}/3\mathbb{Z}$. Our V/F has $\text{cd}(F) \geq 3$ and $V(F) = \emptyset$ and $H^1(F, \text{Pic}(\overline{V})) \neq 0$. As far as we know, Our result would be the first example of computation of the Brauer group of such varieties.

Proof of Theorem 5.1. We recall some notations in §3. We define

$$\alpha = \sqrt[3]{\lambda}, \quad \gamma = \sqrt[3]{\nu}, \quad \alpha' = \sqrt[3]{\mu}, \quad \beta = \alpha\gamma.$$

Moreover we put

$$F' = F(\alpha, \gamma), \quad F'' = F'(\alpha') = F(\alpha, \gamma, \alpha').$$

We have the following exact sequence:

$$0 \rightarrow \text{Br}(V)/\text{Br}(F) \rightarrow H^1(F, \text{Pic}(\overline{V})) \xrightarrow{d^{1,1}} H^3(F, \overline{F}^*).$$

and we know $H^1(F, \text{Pic}(\overline{V})) \cong H^1(F'/F, \text{Pic}(V_{F'})) \cong \mathbb{Z}/3\mathbb{Z}$. Therefore, to prove the theorem, it suffices to show the image of $\phi \in H^1(F'/F, \text{Pic}(V_{F'}))$ in Proposition 3.2 (2) does not vanish in $H^3(F, \overline{F}^*)$.

Before proving this claim, we sketch an outline of its proof. The proof consists of 4 steps. In Step 1, we compute the image of ϕ under the differential

$$d^{1,1}: H^1(F'/F, \text{Pic}(V_{F'})) \rightarrow H^3(F'/F, (F')^*)$$

explicitly. Since the inflation $i_{\overline{F}}^{F'}: H^3(F'/F, F'^*) \rightarrow H^3(F, \overline{F}^*)$ does not necessarily injective, if we prove that $d^{1,1}(\phi) \neq 0$ in $H^3(F'/F, F'^*)$, this is insufficient to prove the theorem. In Step 2, we consider $d^{1,1}(\phi)$ as an element of $H^3(F''/F, \mu_3)$. In Step 3, by computing the residue of $d^{1,1}(\phi)$ along a certain prime divisor D in \mathbb{A}_k^3 and replacing the base field k with the field adding all roots of unity, we reduce the proof to showing that a certain cocycle induced by ϕ is nontrivial in $H^2(k(D), \mu_3)$, where $k(D)$ is the function field of D . Finally, in Step 4, we again compute the residue of the cocycle in Step 3 along a certain prime divisor D' in D and check this is nonzero. These steps complete the proof of the theorem.

Step 1. Let

$$\begin{aligned} \partial: H^1(F'/F, \text{Pic}(V_{F'})) &\rightarrow H^2(F'/F, \mathcal{D}_0), \\ \delta: H^2(F'/F, \mathcal{D}_0) &\rightarrow H^3(F'/F, F'^*) \end{aligned}$$

be connecting homomorphisms induced by the exact sequence in Lemma 3.3 and

$$0 \rightarrow F'^* \rightarrow \text{div}^{-1}(\mathcal{D}_0) \rightarrow \mathcal{D}_0 \rightarrow 0.$$

We have

$$d^{1,1} = \delta \circ \partial: H^1(F'/F, \text{Pic}(V_{F'})) \rightarrow H^3(F'/F, F'^*)$$

by [9], Proposition 6.1. First we compute the cocycle $\partial\phi \in Z^2(F'/F, \mathcal{D}_0)$. Let \mathcal{D} and \mathcal{D}_0 be as in §3. We take

$$0, \quad L(0) - L(2), \quad L(0) - L(1) \in \mathcal{D}$$

as lifts of 0, $[L(0)] - [L(2)]$ and $[L(0)] - [L(1)] \in \text{Pic}(V_{F'})$ respectively, and note that

$$\text{div} \frac{x + \zeta^2 \alpha y}{x} = \text{div} \left(-\frac{\mu f_3 f_4 f_5}{f_1 f_2} \right) \in \mathcal{D}_0.$$

From the construction of the map ∂ , we get the following equations

$$\begin{aligned} \partial\phi(1, 1) &= 0, & \partial\phi(1, s) &= 0, & \partial\phi(1, s^2) &= 0, \\ \partial\phi(s, 1) &= 0, & \partial\phi(s, s) &= \text{div} \frac{z + \zeta \beta t}{x + \zeta^2 \alpha y}, & \partial\phi(s, s^2) &= \text{div} \frac{x + \alpha y}{z + \zeta^2 \beta t}, \\ \partial\phi(s^2, 1) &= 0, & \partial\phi(s^2, s) &= \text{div} \frac{x + \alpha y}{z + \zeta \beta t}, & \partial\phi(s^2, s^2) &= \text{div} \frac{z + \zeta^2 \beta t}{x + \zeta \alpha y}, \\ \partial\phi(t, 1) &= 0, & \partial\phi(t, s) &= 0, & \partial\phi(t, s^2) &= 0, \\ \partial\phi(st, 1) &= 0, & \partial\phi(st, s) &= \text{div} \frac{z + \zeta^2 \beta t}{x + \alpha y}, & \partial\phi(st, s^2) &= \text{div} \frac{x + \zeta \alpha y}{z + \beta t}, \\ \partial\phi(s^2 t, 1) &= 0, & \partial\phi(s^2 t, s) &= \text{div} \frac{x + \zeta \alpha y}{z + \zeta^2 \beta t}, & \partial\phi(s^2 t, s^2) &= \text{div} \frac{z + \beta t}{x + \zeta^2 \alpha y}, \\ \partial\phi(t^2, 1) &= 0, & \partial\phi(t^2, s) &= 0, & \partial\phi(t^2, s^2) &= 0, \\ \partial\phi(st^2, 1) &= 0, & \partial\phi(st^2, s) &= \text{div} \frac{z + \beta t}{x + \zeta \alpha y}, & \partial\phi(st^2, s^2) &= \text{div} \frac{x + \zeta^2 \alpha y}{z + \zeta \beta t}, \\ \partial\phi(s^2 t^2, 1) &= 0, & \partial\phi(s^2 t^2, s) &= \text{div} \frac{x + \zeta^2 \alpha y}{z + \beta t}, & \partial\phi(s^2 t^2, s^2) &= \text{div} \frac{z + \zeta \beta t}{x + \alpha y}, \\ \partial\phi(s^{i_1} t^{j_1}, s^{i_2} t^{j_2}) &= \partial\phi(s^{i_1} t^{j_1}, s^{i_2 - j_2}), \end{aligned}$$

where the indices i_1, i_2, j_1 and j_2 take on any values in $\{0, 1, 2\}$.

Sending this cocycle under δ , we get $\delta\partial\phi$ in $Z^3(F'/F, F'^*)$. If we take

$$1, \quad \frac{x + \zeta^i \alpha y}{z + \zeta^j \beta t}, \quad \frac{z + \zeta^j \beta t}{x + \zeta^i \alpha y} \in \text{div}^{-1}(\mathcal{D}_0)$$

as lifts of 0, $\text{div} \frac{x + \zeta^i \alpha y}{z + \zeta^j \beta t}$ and $\text{div} \frac{z + \zeta^j \beta t}{x + \zeta^i \alpha y} \in \mathcal{D}_0$ respectively, this cocycle is determined by the

following equations:

$$\begin{aligned}
&\delta\partial\phi(t^{j_1}, s^{i_2}t^{j_2}, s^{i_3}t^{j_3}) = 1, \\
&\delta\partial\phi(s^{i_1}t^{j_1}, 1, s^{i_3}t^{j_3}) = 1, \\
&\delta\partial\phi(s^{i_1}t^{j_1}, s^{i_2}t^{j_2}, 1) = 1, \\
&\delta\partial\phi(st^{j_1}, s, s) = 1, \quad \delta\partial\phi(st^{j_1}, s, s^2) = -\mu, \\
&\delta\partial\phi(st^{j_1}, s^2, s) = 1, \quad \delta\partial\phi(st^{j_1}, s^2, s^2) = -\mu^{-1}, \\
&\delta\partial\phi(st^{j_1}, t, s) = 1, \quad \delta\partial\phi(st^{j_1}, t, s^2) = -\mu^{-1}, \\
&\delta\partial\phi(st^{j_1}, st, s) = -\mu^{-1}, \quad \delta\partial\phi(st^{j_1}, st, s^2) = -\mu, \\
&\delta\partial\phi(st^{j_1}, s^2t, s) = -\mu, \quad \delta\partial\phi(st^{j_1}, s^2t, s^2) = 1, \\
&\delta\partial\phi(st^{j_1}, t^2, s) = -\mu, \quad \delta\partial\phi(st^{j_1}, t^2, s^2) = 1, \\
&\delta\partial\phi(st^{j_1}, st^2, s) = -\mu^{-1}, \quad \delta\partial\phi(st^{j_1}, st^2, s^2) = 1, \\
&\delta\partial\phi(st^{j_1}, s^2t^2, s) = 1, \quad \delta\partial\phi(st^{j_1}, s^2t^2, s^2) = 1, \\
&\delta\partial\phi(s^2t^{j_1}, s, s) = -\mu^{-1}, \quad \delta\partial\phi(s^2t^{j_1}, s, s^2) = 1, \\
&\delta\partial\phi(s^2t^{j_1}, s^2, s) = -\mu, \quad \delta\partial\phi(s^2t^{j_1}, s^2, s^2) = 1, \\
&\delta\partial\phi(s^2t^{j_1}, t, s) = 1, \quad \delta\partial\phi(s^2t^{j_1}, t, s^2) = -\mu, \\
&\delta\partial\phi(s^2t^{j_1}, st, s) = 1, \quad \delta\partial\phi(s^2t^{j_1}, st, s^2) = 1, \\
&\delta\partial\phi(s^2t^{j_1}, s^2t, s) = 1, \quad \delta\partial\phi(s^2t^{j_1}, s^2t, s^2) = -\mu^{-1}, \\
&\delta\partial\phi(s^2t^{j_1}, t^2, s) = -\mu^{-1}, \quad \delta\partial\phi(s^2t^{j_1}, t^2, s^2) = 1, \\
&\delta\partial\phi(s^2t^{j_1}, st^2, s) = 1, \quad \delta\partial\phi(s^2t^{j_1}, st^2, s^2) = -\mu, \\
&\delta\partial\phi(s^2t^{j_1}, s^2t^2, s) = -\mu, \quad \delta\partial\phi(s^2t^{j_1}, s^2t^2, s^2) = -\mu^{-1}, \\
&\delta\partial\phi(s^{i_1}t^{j_1}, s^{i_2}t^{j_2}, s^{i_3}t^{j_3}) = \delta\partial\phi(s^{i_1}t^{j_1}, s^{i_2}t^{j_2}, s^{i_3-j_3}),
\end{aligned}$$

where the indices i_1, i_2, i_3, j_1, j_2 and j_3 take on any values in $\{0, 1, 2\}$.

Step 2. Let $i_{\overline{F}}^{F'}$ be the inflation

$$i_{\overline{F}}^{F'} : H^3(F'/F, F'^*) \rightarrow H^3(F, \overline{F}^*).$$

The class $i_{\overline{F}}^{F'} \delta\partial[\phi]$ in $H^3(F, \overline{F}^*)$ is a 3-torsion element, hence by the Kummer sequence, $i_{\overline{F}}^{F'} \delta\partial[\phi]$ comes from $H^3(F, \mu_3)$. Now we find a finite extension K over F such that $i_{\overline{F}}^{F'} \delta\partial[\phi]$ comes from $H^3(K/F, \mu_3)$. In fact, we can take $K = F''$:

Proposition 5.5. *The class $i_{\overline{F}}^{F'} \delta\partial[\phi] \in H^3(\overline{F}/F, \overline{F}^*)$ comes from $H^3(F''/F, \mu_3)$.*

Proof. The exact sequence of $\text{Gal}(F''/F)$ -modules

$$1 \rightarrow \mu_3 \rightarrow F''^* \xrightarrow{3} (F''^*)^3 \rightarrow 1,$$

yields the following commutative diagram

$$\begin{array}{ccccc}
& & H^3(F'/F, F'^*) & & \\
& & \downarrow i_{F''}^{F'} & & \\
H^3(F''/F, \mu_3) & \longrightarrow & H^3(F''/F, F''^*) & \xrightarrow{3} & H^3(F''/F, (F''^*)^3) \\
\downarrow & & \downarrow i_{\overline{F}}^{F''} & & \downarrow \\
H^3(F, \mu_3) & \longrightarrow & H^3(F, \overline{F}^*) & \xrightarrow{3} & H^3(F, \overline{F}^*),
\end{array}$$

where $i_{F'}^{F''}$ and $i_F^{F''}$ are inflations and each row is exact. To prove the claim, it suffices to show $i_{F'}^{F''} \delta \partial[\phi]$ vanishes in $H^3(F''/F, (F''^*)^3)$. Let w be the generator of $\text{Gal}(F''/F')$ defined as §3. The image of $i_{F'}^{F''} \delta \partial[\phi]$ under $3: H^3(F''/F, F''^*) \rightarrow H^3(F''/F, (F''^*)^3)$ is the class of the following cocycle:

$$(s^{i_1} t^{j_1} w^{k_1}, s^{i_2} t^{j_2} w^{k_2}, s^{i_3} t^{j_3} w^{k_3}) \mapsto \delta \partial \phi(s^{i_1} t^{j_1}, s^{i_2} t^{j_2}, s^{i_3} t^{j_3})^3,$$

and what we have to prove is that this cocycle is in $B^3(F''/F, (F''^*)^3)$. Define $\psi \in C^2(F''/F, (F''^*)^3)$ to be:

$$\begin{aligned} \psi(t^{j_1} w^{k_1}, s^{i_2} t^{j_2} w^{k_2}) &= 1, & \psi(s^{i_1} t^{j_1} w^{k_1}, w^{k_2}) &= 1, \\ \psi(st^{j_1} w^{k_1}, sw^{k_2}) &= -\mu^{-1}, & \psi(st^{j_1} w^{k_1}, s^2 w^{k_2}) &= -\mu, \\ \psi(s^2 t^{j_1} w^{k_1}, sw^{k_2}) &= -\mu, & \psi(s^2 t^{j_1} w^{k_1}, s^2 w^{k_2}) &= -\mu^{-1}, \\ \psi(s^{i_1} t^{j_1} w^{k_1}, s^{i_2} t^{j_2} w^{k_2}) &= \psi(s^{i_1} t^{j_1} w^{k_1}, s^{i_2-j_2} w^{k_2}), \end{aligned}$$

where indices i_* , j_* and k_* take on any value in $\{0, 1, 2\}$. Then we can easily see $d\psi = (i_{F'}^{F''} \delta \partial \phi)^3$ in $C^3(F''/F, (F''^*)^3)$ and hence the class of $i_{F'}^{F''} \delta \partial \phi$ vanishes in $H^3(F''/F, (F''^*)^3)$. This completes the proof of Proposition 5.5. \square

By using this cochain ψ , we can construct a cocycle $\Phi \in Z^3(F''/F, \mu_3)$ whose image in $H^3(F, \overline{F}^*)$ is $i_F^{F''} \delta \partial[\phi]$ in a usual manner. As a lift $\tilde{\psi}$ of ψ , we can take the following cochain:

$$\begin{aligned} \tilde{\psi}(t^{j_1} w^{k_1}, s^{i_2} t^{j_2} w^{k_2}) &= 1, & \tilde{\psi}(s^{i_1} t^{j_1} w^{k_1}, w^{k_2}) &= 1, \\ \tilde{\psi}(st^{j_1} w^{k_1}, sw^{k_2}) &= -\alpha'^{-1}, & \tilde{\psi}(st^{j_1} w^{k_1}, s^2 w^{k_2}) &= -\alpha', \\ \tilde{\psi}(s^2 t^{j_1} w^{k_1}, sw^{k_2}) &= -\alpha', & \tilde{\psi}(s^2 t^{j_1} w^{k_1}, s^2 w^{k_2}) &= -\alpha'^{-1}, \\ \tilde{\psi}(s^{i_1} t^{j_1} w^{k_1}, s^{i_2} t^{j_2} w^{k_2}) &= \tilde{\psi}(s^{i_1} t^{j_1} w^{k_1}, s^{i_2-j_2} w^{k_2}). \end{aligned}$$

Then we can take the cocycle $\Phi \in Z^3(F''/F, \mu_3)$ explicitly as follows:

$$(s^{i_1} t^{j_1} w^{k_1}, s^{i_2} t^{j_2} w^{k_2}, s^{i_3} t^{j_3} w^{k_3}) \mapsto \frac{\tilde{\psi}(s^{i_2} t^{j_2} w^{k_2}, s^{i_3} t^{j_3} w^{k_3})}{w^{k_1} \tilde{\psi}(s^{i_2} t^{j_2} w^{k_2}, s^{i_3} t^{j_3} w^{k_3})} \in \mu_3.$$

Step 3. For any prime divisor $D \subset \mathbb{A}_k^3 = \text{Spec } k[\lambda, \mu, \nu]$, we have the following commutative diagram:

$$\begin{array}{ccc} H^3(F''/F, \mu_3) & & \\ \downarrow i_F^{F''} & & \\ H^3(F, \mu_3) & \xrightarrow{\text{res}_D} & H^2(k(D), \mathbb{Z}/3\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^3(F, \mathbb{Q}/\mathbb{Z}(1)) & \xrightarrow{\text{res}_D} & H^2(k(D), \mathbb{Q}/\mathbb{Z}) \\ \downarrow \cong & & \\ H^3(F, \overline{F}^*), & & \end{array}$$

where $F = k(\lambda, \mu, \nu)$ is considered as the function field of \mathbb{A}_k^3 , $k(D)$ is the function field of D , and res_D are residue maps associated to D .

Recall that our goal is to prove the nontriviality of $i_F^{F''} \delta \partial[\phi] \in H^3(F, \overline{F}^*)$. To prove this, by the above diagram, it suffices to show:

$$\text{There exists } D \subset \mathbb{A}_k^3 \text{ such that } \text{res}_D(i_F^{F''} [\Phi]) \neq 0 \in H^2(k(D), \mathbb{Q}/\mathbb{Z}). \quad (5.1)$$

In the sequel, D always denotes the divisor $\{\mu = 0\} \subset \mathbb{A}_k^3$. Let \mathcal{O}_D be the completion of the local ring $k[\lambda, \mu, \nu]_{(\mu)}$ at its maximal ideal and F_D its fractional field. Note that μ is a uniformizer of \mathcal{O}_D and the residue field of \mathcal{O}_D is isomorphic to $k(D) = k(\lambda, \nu)$.

Now we should recall the definition of res_D . There is the canonical isomorphism

$$\iota: \text{Hom}(G_{F_D^{\text{ur}}}, \mu_3) = H^1(F_D^{\text{ur}}, \mu_3) \cong F_D^{\text{ur}*} / (F_D^{\text{ur}*})^3 \cong \mathbb{Z}/3\mathbb{Z},$$

where the middle isomorphism is induced by Kummer sequence and the right one is given by normalized valuation on F_D^{ur} . Then res_D is given by

$$H^3(F, \mu_3) \rightarrow H^3(F_D, \mu_3) \xrightarrow{r} H^2(k(D), \text{Hom}(G_{F_D^{\text{ur}}}, \mu_3)) \xrightarrow{H^2(\iota)} H^2(k(D), \mathbb{Z}/3\mathbb{Z}).$$

For an explicit description of the residue map r , see [6](III, Theorem 6.1).

Now we describe the class $r[i_{\frac{F''}{F}}^{F''} \Phi] \in H^2(k(D), \text{Hom}(G_{F_D^{\text{ur}}}, \mu_3))$ explicitly. By the definition of r and the fact $i_{\frac{F''}{F}}^{F''} \Phi$ originally comes from the cocycle Φ of $\text{Gal}(F''/F)$, we would naturally expect that $r i_{\frac{F''}{F}}^{F''} \Phi$ also comes from the cocycle of the Galois group of residue fields of F''/F along to D . In fact, we find that it is true.

Before stating the claim, we introduce some field extensions. Let $k(D)'$, F_D'' , F_D' be the same notation as in §3. Moreover, by abuse of notation, we denote the elements in $\text{Gal}(F_D''/F_D)$ corresponding to s , t and $w \in \text{Gal}(F''/F)$ as the same symbols. To make our situation clear, we give the following diagram of field extensions:

$$\begin{array}{ccccc}
 & & F_D'' & \xrightarrow{\text{residue field}} & k(D)' \\
 & \nearrow & \downarrow 3 \text{ ramified} & & \parallel \\
 F'' & & F_D' & \xrightarrow{\quad \quad \quad} & k(D)' \\
 \downarrow 3 & \nearrow & \downarrow 9 \text{ unramified} & & \downarrow 9 \\
 F' & & F_D & \xrightarrow{\quad \quad \quad} & k(D) \\
 \downarrow 9 & \nearrow \text{completion} & & & \\
 F & & & &
 \end{array}$$

The claim is:

Lemma 5.6. *If we define the cochain*

$$\overline{r\Phi} \in C^2(k(D)'/k(D), \text{Hom}(\text{Gal}(F_D''/F_D'), \mu_3))$$

as

$$\overline{r\Phi}(\overline{s}^{i_1} \overline{t}^{j_1}, \overline{s}^{i_2} \overline{t}^{j_2})(w^k) := \Phi(w^k, s^{i_1} t^{j_1}, s^{i_2} t^{j_2}),$$

where \overline{s} and \overline{t} is the image of s and t under the natural map

$$\text{Gal}(F_D''/F_D) \rightarrow \text{Gal}(k(D)'/k(D)),$$

then $\overline{r\Phi}$ is a cocycle and its image under the map

$$i_{\frac{k(D)'}{k(D)}}^{k(D)'}: H^2(k(D)'/k(D), \text{Hom}(\text{Gal}(F_D''/F_D'), \mu_3)) \rightarrow H^2(k(D), \text{Hom}(G_{F_D^{\text{ur}}}, \mu_3))$$

is $r[i_{\frac{F''}{F}}^{F''} \Phi]$.

Proof. we can prove that $\overline{r\Phi}$ is a cocycle by a straightforward calculation. The latter claim is easy to check by the definition of r and the proof is left to the reader. \square

By using natural isomorphisms ι and

$$\mathrm{Hom}(\mathrm{Gal}(F_D''/F_D'), \mu_3) \cong \mathbb{Z}/3\mathbb{Z}; \quad (w \mapsto \zeta) \mapsto 1,$$

we rewrite $i_{\frac{k(D)'}{k(D)}}^{k(D)'}$ simply as follows:

$$H^2(k(D)'/k(D), \mathbb{Z}/3\mathbb{Z}) \rightarrow H^2(k(D), \mathbb{Z}/3\mathbb{Z}).$$

For a field K of characteristic 0, we denote \widetilde{K} by $\bigcup_{n>0} K(\zeta_n)$, where ζ_n is a primitive n -th root of unity. Noting that $k(D)' = k(D)(\alpha, \gamma)$ and that α and γ are transcendental over k , we have $k(D)' \cap \widetilde{k(D)} = k(D)$ and therefore

$$\mathrm{Gal}(\widetilde{k(D)'} / \widetilde{k(D)}) \xrightarrow{\cong} \mathrm{Gal}(k(D)' / k(D)).$$

We fix an isomorphism $\mathbb{Q}/\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}(1)$ as trivial $\widetilde{k(D)}$ -modules. Then we have the following commutative diagram:

$$\begin{array}{ccc} H^2(k(D)'/k(D), \mathbb{Z}/3\mathbb{Z}) & \longrightarrow & H^2(k(D), \mathbb{Q}/\mathbb{Z}) \\ \cong \downarrow & & \downarrow \\ H^2(\widetilde{k(D)'} / \widetilde{k(D)}, \mathbb{Z}/3\mathbb{Z}) & \longrightarrow & H^2(\widetilde{k(D)}, \mathbb{Q}/\mathbb{Z}) \\ \cong \downarrow & & \downarrow \cong \\ H^2(\widetilde{k(D)'} / \widetilde{k(D)}, \mu_3) & \longrightarrow & H^2(\widetilde{k(D)}, \mathbb{Q}/\mathbb{Z}(1)) \\ \downarrow & & \downarrow \cong \\ H^2(\widetilde{k(D)}, \mu_3) & \longrightarrow & H^2(\widetilde{k(D)}, \overline{k(D)}^*). \end{array}$$

Since the bottom map in the above diagram is injective by Hilbert's Theorem 90, in order to prove the claim (5.1), it suffices to show:

$$[\overline{r\Phi}] \in H^2(k(D)'/k(D), \mathbb{Z}/3\mathbb{Z}) \text{ does not vanish in } H^2(\widetilde{k(D)}, \mu_3).$$

Step 4. For simplicity, we put $E = \widetilde{k(D)} = \widetilde{k}(\lambda, \nu)$ and $E' = \widetilde{k(D)'} = E(\alpha, \gamma)$. we define the cocycle $\Psi \in Z^2(E'/E, \mu_3)$ as follows:

$$\Psi(t^{j_1}, s^{i_2} t^{j_2}) = 1, \quad \Psi(st^{j_1}, s^{i_2} t^{j_2}) = \begin{cases} 1 & j_2 = 0 \\ \zeta^2 & j_2 = 1 \\ \zeta & j_2 = 2, \end{cases} \quad \Psi(s^2 t^{j_1}, s^{i_2} t^{j_2}) = \begin{cases} 1 & j_2 = 0 \\ \zeta & j_2 = 1 \\ \zeta^2 & j_2 = 2. \end{cases}$$

We can easily check that $[\Psi]$ is the image of $[\overline{r\Phi}] \in H^2(k(D)'/k(D), \mathbb{Z}/3\mathbb{Z})$ under the isomorphism $H^2(k(D)'/k(D), \mathbb{Z}/3\mathbb{Z}) \xrightarrow{\cong} H^2(E'/E, \mu_3)$ in the above diagram.

What we have to show is that the image of $[\Psi] \in H^2(E'/E, \mu_3)$ under

$$i_{\frac{E'}{E}}^{E'} : H^2(E'/E, \mu_3) \rightarrow H^2(E, \mu_3)$$

is nonzero. This is a consequence of the following:

Proposition 5.7. *Put $D' = \{\nu = 0\} \subset \mathbb{A}_k^2$. The image of $i_{\frac{E'}{E}}^{E'}[\Psi]$ under the residue map*

$$\mathrm{res}_{D'} : H^2(E, \mu_3) \rightarrow H^1(k(D'), \mathbb{Z}/3\mathbb{Z})$$

is nonzero.

Proof. We fix notations. Let $\mathcal{O}_{D'}$ be the completion of the local ring $k[\lambda, \nu]_{(\nu)}$ at its maximal ideal and $E_{D'}$ its fractional field. Note that ν is a uniformizer of $\mathcal{O}_{D'}$ and the residue field of $\mathcal{O}_{D'}$ is isomorphic to $k(D') = k(\lambda)$. Let $E'_{D'}$ be the same notation as in §3. By abuse of notation, we denote the elements in $\text{Gal}(E'_{D'}/E_{D'})$ corresponding to s and $t \in \text{Gal}(E'/E)$ as the same symbols. To make our situation clear, we give the following diagram of field extensions:

$$\begin{array}{ccccc}
 & & E'_{D'} & \xrightarrow{\text{residue field}} & k(D')(\alpha) \\
 & \nearrow & \downarrow 3 \text{ ramified} & & \parallel \\
 E' & & E_{D'}(\alpha) & \xrightarrow{\quad} & k(D')(\alpha) \\
 \downarrow 3 & \nearrow & \downarrow 3 \text{ unramified} & & \downarrow 3 \\
 E(\alpha) & & E_{D'} & \xrightarrow{\quad} & k(D') \\
 \downarrow 3 & \nearrow \text{completion} & & & \\
 E & & & &
 \end{array}$$

Now $\text{res}_{D'}$ is given by

$$H^2(E, \mu_3) \rightarrow H^2(E_{D'}, \mu_3) \xrightarrow{\tau} H^1(k(D'), \text{Hom}(G_{E_{D'}^{\text{ur}}}, \mu_3)) \xrightarrow{\cong} H^1(k(D'), \mathbb{Z}/3\mathbb{Z}).$$

We also have a similar result to Lemma 5.6:

Lemma 5.8. *If we define the cochain*

$$\overline{r\Psi} \in C^1(k(D')(\alpha)/k(D'), \text{Hom}(\text{Gal}(E'_{D'}/E_{D'}(\alpha)), \mu_3))$$

as

$$\overline{r\Psi}(\bar{t}^j)(s^i) := \Psi(s^i, t^j),$$

where \bar{t} is the image of t under the natural map

$$\text{Gal}(E'_{D'}/E_{D'}) \rightarrow \text{Gal}(k(D')(\alpha)/k(D')),$$

then $\overline{r\Psi}$ is a cocycle and its image under the map

$$i_{\frac{k(D')(\alpha)}{k(D')}} : H^1(k(D')(\alpha)/k(D'), \text{Hom}(\text{Gal}(E'_{D'}/E_{D'}(\alpha)), \mu_3)) \rightarrow H^1(k(D'), \text{Hom}(G_{E_{D'}^{\text{ur}}}, \mu_3))$$

is $ri_E^{E'}[\Psi]$.

Proof. The claim follows from similar calculations in Lemma 5.6. The details are left to the reader. \square

We now go back to the proof of Proposition 5.7. We know that $i_{\frac{k(D')(\alpha)}{k(D')}}^{k(D')(\alpha)}$ is injective. Moreover, we can easily check $\overline{r\Psi} \neq 0$ by definition. Therefore $\text{res}_{D'}(i_E^{E'}[\Psi]) \neq 0$, which completes the proof of Proposition 5.7. \square

Theorem 5.1 is a consequence of Proposition 5.7. \square

Next we give a proof of Proposition 5.2. Before proving the proposition, we note the following.

Lemma 5.9. *Let S_0, S_1 and S_2 be infinite subsets of k^* . Then $S_0 \times S_1 \times S_2$ is Zariski dense in $(\mathbb{G}_{m,k})^3$.*

Proof. It suffices to show for any sufficient small open subscheme W in $(\mathbb{G}_{m,k})^3$, the intersection $(S_0 \times S_1 \times S_2) \cap W$ is non-empty. Since all open subschemes $U_0 \times U_1 \times U_2 \subset (\mathbb{G}_{m,k})^3$ form an open base, where U_i run through affine open subschemes in $\mathbb{G}_{m,k}$, we may assume that W is of the form $U \times U \times U$ for an affine open subscheme $U \subset \mathbb{G}_{m,k}$. Hence it suffices to prove that $S_0 \cap U \neq \emptyset$ for any sufficiently small affine open $U \subset \mathbb{G}_{m,k}$. We may take U as an affine open subscheme of the form:

$$U = \text{Spec } k[\lambda^\pm]_f, \quad 0 \neq f \in k[\lambda].$$

Since S_0 is infinite, there exists $\lambda_0 \in S_0$ such that $f(\lambda_0) \neq 0$. Then $\lambda_0 \in S_0 \cap U$ and in particular $S_0 \cap U$ is non-empty. \square

Proof of Proposition 5.2. (1) \Rightarrow (2). This is a trivial implication.

(2) \Rightarrow (3). We prove the contrapositive statement. If we assume $\dim_{\mathbb{F}_3} k^*/(k^*)^3 = 1$, we can take $v \in k^* \setminus (k^*)^3$. Then the equation of diagonal cubic surfaces is essentially equal to one of the following:

$$\begin{aligned} x^3 + y^3 + z^3 + t^3 &= 0, & x^3 + y^3 + z^3 + vt^3 &= 0, & x^3 + y^3 + z^3 + v^2t^3 &= 0, \\ x^3 + y^3 + vz^3 + vt^3 &= 0, & x^3 + y^3 + vz^3 + v^2t^3 &= 0, \end{aligned}$$

all of which have a k -rational point. We also see by Proposition 3.1 that $H^1(k, \text{Pic}(\overline{V_P})) \cong 0$ or $(\mathbb{Z}/3\mathbb{Z})^2$ for all $P \in (\mathbb{G}_{m,k})^3$, and therefore

$$\forall P \in (\mathbb{G}_{m,k})^3, \text{Br}(V_P)/\text{Br}(k) \cong 0 \text{ or } (\mathbb{Z}/3\mathbb{Z})^2$$

by Lemma 2.1. We can also prove the case $\dim_{\mathbb{F}_3} k^*/(k^*)^3 = 0$ in a similar way. Hence we have $\mathcal{P}_k = \emptyset$.

(3) \Rightarrow (1). We first construct a subset \mathcal{P} of \mathbb{A}_k^3 satisfying the following three conditions:

- (i) \mathcal{P} is Zariski dense in \mathbb{A}_k^3 ;
- (ii) $P \in \mathcal{P} \Rightarrow V_P(k) \neq \emptyset$;
- (iii) $P \in \mathcal{P} \Rightarrow H^1(k, \text{Pic}(\overline{V_P})) \cong \mathbb{Z}/3\mathbb{Z}$.

Since $\dim_{\mathbb{F}_3} k^*/(k^*)^3 \geq 2$, we can take two linearly independent elements v_1 and v_2 . Now we define \mathcal{P} as

$$\mathcal{P} = S_0 \times S_1 \times S_2, \quad S_0 = (k^*)^3, \quad S_1 = v_1(k^*)^3, \quad S_2 = v_2(k^*)^3.$$

We show that \mathcal{P} satisfies the above three conditions. First, by Lemma 5.9 and the assumption that $(k^*)^3$ is infinite, the condition (i) holds. Secondly, for $P = (\lambda_0, \mu_0, \nu_0) \in \mathcal{P}$, we can take $\lambda'_0 \in k^*$ such that $(\lambda'_0)^3 = \lambda_0$, and V_P has a k -rational point $(\lambda'_0 : -1 : 0 : 0)$. Hence the condition (ii) holds. Finally, by the choice of v_1 and $v_2 \in k^*$ and Proposition 3.1, we can see that the condition (iii) holds.

Conditions (ii), (iii) and Lemma 2.1 imply $\text{Br}(V_P)/\text{Br}(k) \cong \mathbb{Z}/3\mathbb{Z}$ for all $P \in \mathcal{P}$ and therefore we complete the proof of (1) \Rightarrow (3).

This completes the proof of Proposition 5.2. \square

Proof of Corollary 5.3. We would have an element $e \in \text{Br}(V)$ satisfying the property stated in Corollary 5.3:

there exists a dense open subset $W \subset (\mathbb{G}_{m,k})^3$ such that $\text{sp}(e; \cdot)$ is defined on $W(k) \cap \mathcal{P}_k$ and for all $P \in W(k) \cap \mathcal{P}_k$, $\text{sp}(e; P)$ is a generator of $\text{Br}(V_P)/\text{Br}(k)$.

By Theorem 5.1, we have

$$\text{Br}(V)/\text{Br}(F) = 0$$

and hence there exists an element $e' \in \text{Br}(F)$ such that $\pi_F^* e' = e$. We have the isomorphism

$$\varinjlim_i \text{Br}(S_i) = \text{Br}(F),$$

where (S_i) is the projective system of the non-empty open affine subschemes in \mathbb{A}_k^3 , and there exists a non-empty affine open subscheme S and $\tilde{e}' \in \text{Br}(S)$ such that \tilde{e}' is a lift of e' and $\mathcal{V} \times_{\mathbb{A}_k^3} S$ is smooth over S . Since \mathcal{P}_k is a Zariski dense set in $(\mathbb{G}_{m,k})^3$ by Proposition 5.2 and $S \cap W$ is a non-empty Zariski open set in $(\mathbb{G}_{m,k})^3$, there exists a point $P \in (S \cap W)(k) \cap \mathcal{P}_k$. For this point P , we have the following commutative diagram:

$$\begin{array}{ccccc} \text{Br}(V_P) & \xleftarrow{P^*} & \text{Br}(\mathcal{V} \times_{\mathbb{A}_k^3} S) & \xhookrightarrow{\quad} & \text{Br}(V) \\ \uparrow \pi_P^* & & \uparrow \pi_S^* & & \uparrow \pi_F^* \\ \text{Br}(k) & \xleftarrow{P^*} & \text{Br}(S) & \xhookrightarrow{\quad} & \text{Br}(F) \end{array}$$

and hence we can take $\pi_S^* \tilde{e}'$ as a lift of e . Then we get

$$\text{sp}(e; P) = P^*(\pi_S^* \tilde{e}') = \pi_P^* P^* \tilde{e}' \in \pi_P^* \text{Br}(k).$$

This means that $\text{sp}(e; P)$ is zero in the group $\text{Br}(V_P)/\text{Br}(k)$, which contradicts that $\text{sp}(e; P)$ is a generator of $\text{Br}(V_P)/\text{Br}(k) \cong \mathbb{Z}/3\mathbb{Z}$. Therefore we see that there is no such element e , and complete the proof of Corollary 5.3. \square

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